International Journal of Theoretical Physics, Vol. 35, No. 8, 1996

Electric Energy Calculations in General Relativity

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Received October 9, 1995

By means of pseudotensor calculations in general relativity, we calculate the electric energy in two particular cases for the Reissner–Nordström metric.

1. INTRODUCTION

In this paper we perform energy-momentum pseudotensor calculations for the Reissner–Nordström metric (Adler *et al.*, 1975) and obtain energy values in two cases: (i) the energy of a charged point mass when the charge has the same absolute value as the "geometric mass"; (ii) the energy of a point charge in the particular case where the "geometric mass" is negligible in relation to the electric charge.

These two cases show interesting features and are a first step toward the study of the general case of a charge Q of mass M.

2. ENERGY OF A CHARGED POINT MASS

The field of a charged point mass is given by the well-known Reissner–Nordström relation (Adler *et al.*, 1975):

$$ds^2 = g_{00}c^2\, dt^2 - g_{00}^{-1}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.1}$$

where

$$g_{00} = 1 - \frac{2m}{r} + \frac{\alpha}{r^2} \equiv 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{4\pi c^4 r^2} \tag{2.2}$$

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Papapetrou (1945–8), Bonnor (1965), and others studied the particular case when

$$|Q| = m \tag{2.3}$$

For this case, we have, in isotropic coordinates, the result

$$ds^2 = \left(1 + \frac{m}{\rho}\right)^{-2} c^2\, dt^2 - \left(1 + \frac{m}{\rho}\right)^2 (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2) \tag{2.4}$$

When we have a static metric in isotropic coordinates and the system is "Lorentzian at infinity" (Adler *et al.*, 1975) we may apply the Einstein pseudotensor calculation presented by Adler *et al.* (1975), which gives for the energy of the system the formula

$$P_0 = -\frac{c^2 R}{G} \int_{S^2} g_{00}^{1/2} (-g_{\rho\rho})^{3/2} g^{\rho\rho}_{,\rho}\, d\rho \qquad (\rho = R) \tag{2.5}$$

Plugging the metric elements from (2.4) into (2.5), we get the final result

$$P_0 = \frac{2RC^2}{G} \ln\left(1 + \frac{GM}{Rc^2}\right) \tag{2.6}$$

This result means that, when $R \to \infty$, $P_0 \to 2M$. This has to do with the fact that the charge effects sum up with the gravitational mass effects. In the next section we study the Nordström–Reissner metric when, instead of hypothesis (2.3), one makes the assumption that the mass M is negligible in comparison with the charge Q effects. I intend to work on the general case in another article. The main difficulty arises from the requirement of isotropic coordinates "Lorentzian at infinity" in order to apply (2.5). Note that Raychaudhuri and Som (1968) found also that electromagnetic charges alter the total energy of a system.

3. ELECTRIC ENERGY OF A POINT CHARGE

Again, the field of a charged mass point is given by the well-known Reissner–Nordström metric (Adler *et al.*, 1975)

$$ds^2 = g_{00} c^2\, dt^2 - g_{00}^{-1}\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.1}$$

where

$$g_{00} = 1 - \frac{2GM}{c^2 r} + \frac{CQ^2}{4\pi c^4 r^2} \tag{3.2}$$

where G and Q are the gravitational constant and the electric charge, respectively, and M stands for the mass at the origin.

By the method outlined by Adler *et al.* (1975), one can transform this metric to the isotropic form

$$ds^2 = A(\rho)c^2dt^2 - B(\rho)\, d\sigma^2 \tag{3.3}$$

where

$$d\sigma^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.4}$$

On equating (3.3) and (3.1), we find, when the mass M is negligible,

$$ds^2 = \frac{(1 + \alpha/\rho^2)^2}{1 - \alpha/\rho^2}\, c^2\, dt^2 - \left(1 - \frac{\alpha}{\rho^2}\right)^2 d\sigma^2 \tag{3.5}$$

with

$$\alpha = \frac{GQ^2}{16\pi c^4} \tag{3.6}$$

As we can see, the metric (3.5) is Lorentzian when $\rho \to \infty$. In this case, we may apply the Einstein pseudotensor theory and get a constant exact result for the energy P_0 of the field.

Adler *et al.* (1975) show that for the static isotropic metric, as in our case, the following formula applies:

$$P_0 = -\frac{c^4}{8\pi G} \int_{S^2} \sqrt{-g}\, g^{11}|_j n_j\, dS \tag{3.7}$$

where n_j and dS are to be understood in the sense required by classical integral calculus, when the Gauss theorem is applied to a sphere of radius $\rho = R$.

By performing the calculation in (3.7) with the metric (3.5), we find

$$P_0 = \frac{c^4R}{G}\left[\ln(1 - \alpha R^{-2})^{-2\alpha} + \left(1 - \frac{\alpha}{R^2}\right)\right] \tag{3.8}$$

When $R \to \infty$, we find the result

$$\lim_{R\to\infty} P_0 = \infty \tag{3.9}$$

which is unrealistic because we have neglected the mass of the charge, which appears in (3.2).

We obviously have to impose

$$R > \sqrt{\alpha} = R_0 \tag{3.10}$$

A similar effect is found for the Schwarzschild metric, where

$$P_0^S = Mc^2\left(1 - \frac{GM}{2c^2R}\right) \tag{3.11}$$

and

$$R_0^S = \frac{GM}{2c^2} \tag{3.12}$$

We can say that the entire energy is due to the contributions of the field outside of the radius R_0 in the isotropic line element.

For an electron, we have

$$R_0 \approx 10^{-59} \text{ cm}$$

This value is much smaller than the Planck distance

$$L_{\text{Pl}} \cong 10^{-33} \text{ cm}$$

It should be remarked that the corresponding Schwarzschild case for an electron leads to the value

$$R_0^S \approx 10^{-55} \text{ cm}$$

It is obvious that R_0 and R_0^S have negligible values as far as classical general relativity is concerned.

4. FINAL COMMENTS

Our results now should be extended to the full general case where any charge and any mass generate the field energy. This is not simple, because isotropic coordinates are difficult to find for this case. We intend to work on this case in the future.

ACKNOWLEDGMENTS

I thank Prof. M. M. Som for encouragement on this and other papers, and Prof. Ronald Adler for reading a draft of this paper and kindly making comments. This work was financially supported by FAPESP, Brazil.

NOTE ADDED IN PROOF

K. S. Virbhadra, *Phys. Rev. D42* 2919 (1990), obtained different results under different assumptions. I comment on these results in Berman (1996), submitted for publication.

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